Abstract Algebra I
Problem Set 4: Cosets

Exam style questions are marked with (*).

Equivalence relations

Question 1. Prove that an equivalence relation $\sim$ on a set S partitions the set into distinct subsets S_i, called *equivalence classes*, with $a \sim b$ if and only if a and b belong to the same subset S_i.

One can then lift a binary operation $*$ on S to a binary operation on equivalence classes of S.

Question 2. Let $*$ be a binary operation on S and let $\sim$ be an equivalence relation on S. Denote the equivalence class of $x \in S$ by $[x]$. Let $[S]$ denote the set of all equivalence classes of $\sim$ in S. Let

$$*_\sim : [S] \times [S] \to [S] : \big([a], [b]\big) \mapsto [a * b].$$

Show that this map is well defined and therefore a binary operation. In particular, show that, if $a \sim a'$ and $b \sim b'$ then

$$*_\sim\big([a], [b]\big) = *_\sim\big([a'], [b']\big).$$

Cosets

Question 3. (*) Prove Lagrange's theorem.

Normal groups

Question 4. Prove that a subgroup of index 2 is always normal, i.e. if $|G : H| = 2$ then $H \trianglelefteq G$.

Question 5. Consider a regular tetrahedron with vertices $v_1, \ldots, v_4$, with group of symmetric transformations S_4 and subgroup of symmetric rotations A_4. Consider a fixed vertex, say v_1 and consider the subsets of S_4 and A_4 respectively that *fix* v_1, that is to say, do not change the labelling of this vertex. These subsets are called the *stabilisers of* v_1.

- Prove that the stabiliser of v_1 in S_4 is a subgroup of S_4. Find its left and right cosets. Is it normal?
- Prove that the stabiliser of v_1 in A_4 is a subgroup of S_4. Find its left and right cosets. Is it normal?
- Find a normal subgroup of A_4.

Question 6. The 5-cell is the 4-dimensional simplex, the simplest shape one can construct of same length edges in 4-dimensions. Let G be the group of symmetry preserving rotations of the 5-cell (A_5). Let H be the stabiliser of a single vertex, $H < G$. List the cosets of H. Is H normal in G?